Review of “Ramsey-type results for ordered hypergraphs”, by Martin Balko

In this thesis, the author introduces and explores an ordered version of Ramsey numbers, proving a number of surprising results highlighting the differences between this new notion and the ordinary Ramsey number, solving several conjectures stated by this reviewer together with Fox, Lee and Sudakov, and applying the results to a number of geometric problems. I will give a brief summary of the main results below, but I should first state that I think the thesis is a very strong one, demonstrating a marked ability for independent and creative scientific work, and is in my opinion more than sufficient for the award of a doctorate.

To describe the main results, we first need some definitions. An *ordered hypergraph* $\mathcal{H}$ is a pair $(H, <)$, where H is a hypergraph and $<$ is a total ordering of its vertex set. An ordered hypergraph $\mathcal{H} = (H, <_1)$ is a said to be an *ordered subhypergraph* of $\mathcal{G} = (G, <_2)$ if H is a subhypergraph of G and $<_1$ is a suborder of $<_2$. The central notion of this thesis is the *ordered Ramsey number* of an ordered k-uniform hypergraph $\mathcal{H}$, defined to be the smallest N such that any 2-colouring of the edges of the complete ordered k-uniform hypergraph $\mathcal{K}_N^{(k)}$ contains a monochromatic ordered copy of $\mathcal{H}$.

A brief summary of the contents by chapter is now as follows:

1. The author introduces the notion of ordered Ramsey numbers and proves a number of exact results concerning the ordered Ramsey numbers of some simple ordered graphs, including stars, monotone paths, alternating paths, and monotone cycles.

2. The author studies the growth rate of ordered Ramsey numbers of graphs in a more asymptotic sense. The main results of this chapter include proofs that:

   (a) there are ordered matchings for which the ordered Ramsey number is superpolynomial in the number of vertices;

   (b) if an ordered matching has interval chromatic number two then the ordered Ramsey number is at most quadratic in the number of vertices and this result is almost sharp;

   (c) if an ordered graph has bounded degeneracy and bounded interval chromatic number then its ordered Ramsey number is at most polynomial in the number of vertices;

   (d) if an ordered graph has bounded bandwidth then its ordered Ramsey number is at most polynomial in the number of vertices;

   (e) there are 3-regular graphs such that the ordered Ramsey number is superlinear in the number of vertices for every ordering, while every graph of maximum degree 2 has an ordering in which the ordered Ramsey number is linear.

   It is worth noting that the results stated as (a), (b) and (c) above were obtained independently by Balko, Cibulka, Král, and Kynčl and also by this reviewer together with Fox, Lee, and Sudakov.

The results of (d) and (e) were obtained later by Balko, Cibulka, Král, and Kynčl and by Balko, Jelínek, and Valtr and answered open problems stated by myself, Fox, Lee, and Sudakov.

3. The author discusses a number of applications of ordered Ramsey numbers to geometric problems, including a computer-assisted disproof of a conjecture of Peters and Szekeres which relates to finding convex point sets in the plane (but which may be restated as an ordered Ramsey-type problem) and a quasipolynomial upper bound for the convex geometric Ramsey number of outerplanar graphs.

4. The author proves a tight lower bound for the monotone crossing number of K_n, that is, the minimum number of pairs of crossing edges given that the graph is drawn in the xy-plane so that every edge intersects each vertical line exactly once. This result may also be placed in an alternative context which allows it to be regarded as a Ramsey multiplicity problem for ordered hypergraphs. Finally, the author uses simulated annealing techniques to improve the lower bound for the pseudolinear crossing number of K_n.

In my opinion, the study of ordered Ramsey numbers is an interesting and exciting new development in extremal combinatorics and this thesis is an integral part of this development, laying the groundwork for the further study of ordered Ramsey numbers and pointing the way towards possible applications in geometric combinatorics and beyond. It is often said that the health of a mathematical area is determined by its open problems. This being the case, the study of ordered Ramsey numbers, which has already suggested a wealth of interesting and difficult problems, should be in fine fettle for some time to come.

In conclusion, I believe the thesis to be of very high quality. It contains a large number of original results, all of which are correct, new and interesting, and engages with difficult problems at the frontiers of combinatorial research. The thesis has also been written with great care and diligence and I warmly recommend that the candidate be awarded a PhD on the basis of this work.

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