Problems in Discrete Geometry

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Overview

- Three different problems from discrete geometry
- Common theme: solution is based on linear algebra or algebraic methods

1. Multilevel polynomial partitions and simplified range searching
2. Lower bounds on geometric Ramsey numbers
3. k-reptile simplices in $\mathbb{R}^3$ and $\mathbb{R}^4$

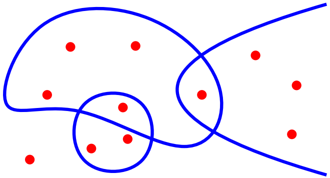
Geometric divide-and-conquer method

- useful for solving numerous problems in discrete and computational geometry
- suitable partition of space is used to subdivide a geometric problem into simpler subproblems
- subproblems usually solved recursively

Examples of such partitions

- cuttings . . . . Clarkson '87; Haussler, Welzl'87
- simplicial partitions . . . . Matoušek '92; Chan'12
- **polynomial** partitions . . . . Guth, Katz '10
  - –**multilevel** version . . . . Matoušek, **ZP** '15

- $P \dots n$ points in $\mathbb{R}^d$
- $r > 1$
- nonzero polynomial f is r-partitioning for P if
  $\bigcup P_i \subset P, \ \dots P_i \subset$ cell $=$ connected component of $\mathbb{R}^d \setminus Z(f)$
  $|P_i| \leq n/r$



$Z(f) = \{x \in \mathbb{R}^d \colon f(x) = 0\}$

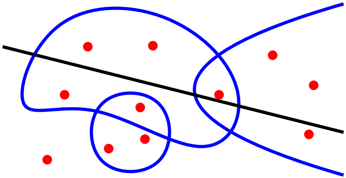
Theorem (Guth, Katz, '10)

For every $r > 1$, every finite point set $P \subset \mathbb{R}^d$ admits an r-partitioning polynomial of degree $O(r^{1/d})$.

- solution to famous Erdős Distinct Distance problem

Polynomial partition

- n points in $\mathbb{R}^d$
- f ... an r-partitioning polynomial for P
- any hyperplane crosses $\leq O(r^{1-1/d})$ cells of $\mathbb{R}^d \setminus Z(f)$.



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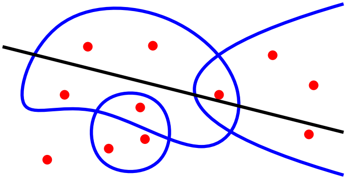
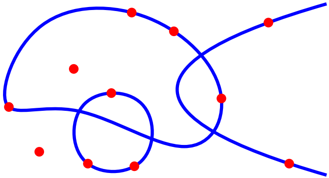


- Barone, Basu'12: any k-dim variety X defined by constant-degree polynomials crosses $\leq O(r^{k/d})$ cells

Main problem

What to do with the exceptional set $P^* := P \cap Z(f)$?



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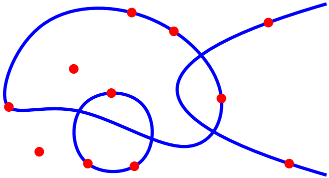

In many applications perturbation of points does not work.

Multilevel polynomial partition

- Natural idea: partition P^* further by another polynomial g s.t. $Z(f, g) := Z(f) \cap Z(g)$ has $\dim \leq d - 2$

- If $Z(f, g)$ again contains many points of P^*, partition them further by a third polynomial h with $\dim Z(f, g, h) \leq d - 3$, and so on.

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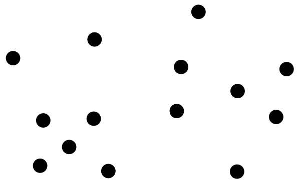

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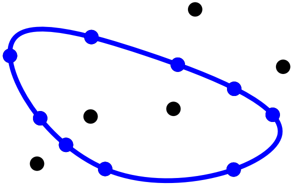

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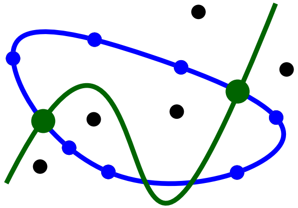

Second partitioning polynomial

Theorem (Kaplan, Matoušek, **ZP**, Sharir'11; Zahl'11)

- $r > 1$
- $f \in \mathbb{R}[x_1, \ldots, x_d]$ *an irreducible poly of degree* $D \geq 1$
- $P \ldots n$ *points in* $Z(f)$

$\Rightarrow$ *There exists an* r*-partitioning polynomial* g *for* P *s.t.*

- $\deg g = O\left(D + \left(\frac{r}{D}\right)^{1/(d-1)}\right)$
- g *is co-prime with* f

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Application: number of unit distances determined by n points in $\mathbb{R}^3$ is $O(n^{3/2})$

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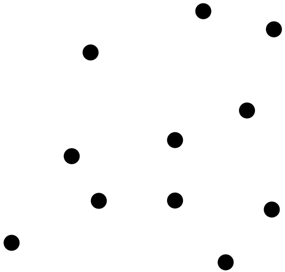

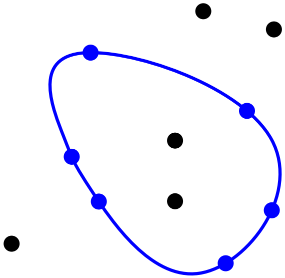

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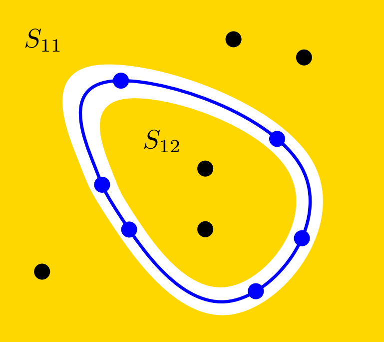

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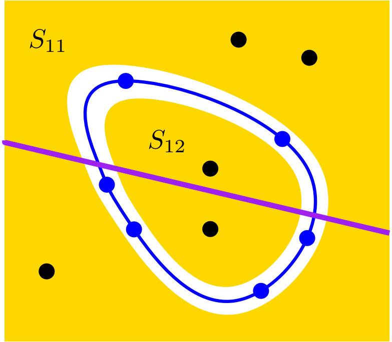

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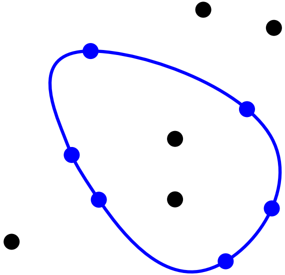

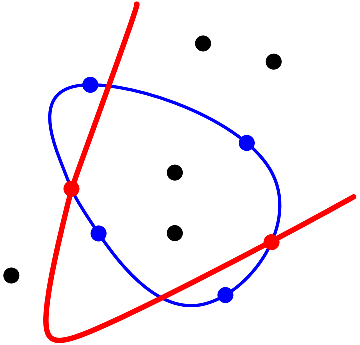

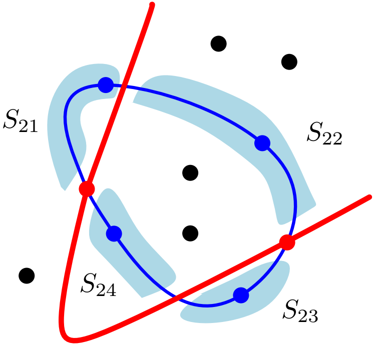

- $|P_{2j}| \leq n/r_2$

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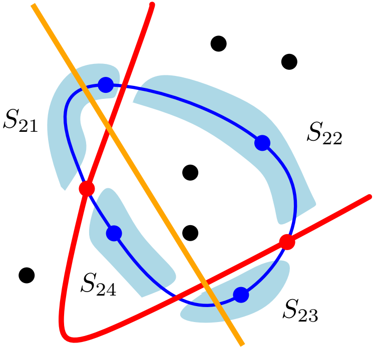

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Theorem (Matoušek, **ZP**, '15)

For every integer $d > 1$ there is a constant K such that:
Given an n-point set $P \subset \mathbb{R}^d$ and $r > 1$,
there are numbers $r_1, r_2, \ldots, r_d \in [r, r^K]$, a partition

$$P = P^* \cup \bigcup_{i=1}^{d} \bigcup_{j=1}^{t_i} P_{ij}$$

of P into disjoint subsets, and for every i, j, a connected set $S_{ij} \subseteq \mathbb{R}^d$ containing P_{ij}, s.t. $|P_{ij}| \leq n/r_i$ for all i, j, $|P^*| \leq r^K$.

Furthermore, if X is a k-dim variety in $\mathbb{R}^d$ defined by polynomials of constant degree then, for every $i = 1, \ldots, d$, the number of the S_{ij} crossed by X is bounded by $O\left(r_i^{1-1/(k+1)}\right)$.

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Application: Semialgebraic range-searching

The partition can be computed effectively in Real RAM model of computation:

- P^*, P_{ij}, and S_{ij} in time $O\left(nr^C\right)$ ... $C = C(d)$ is a constant
- each S_{ij} is a semialgebraic set defined by at most $O\left(r^C\right)$ polynomial inequalities of maximum degree $O\left(r^C\right)$

Consequence: a simpler algorithm for semialgebraic range-searching – technically and conceptually

Semialgebraick range-searching

- P – a set of n points in $\mathbb{R}^d$
  $\Gamma_{d,D,s}$ – a family of semialgebraic sets in $\mathbb{R}^d$ (**“ranges“**)
  defined by $\leq s$ polynomial inequalities of degree $\leq D$ each

  e.g. set of axis-parallel boxes, balls, simplices . . .
- $\Gamma_{d,D,s}$-range searching – preprocessing P into a data structure so that the number of points of P lying in a query range $\gamma \in \Gamma_{d,D,s}$ can be counted efficiently
- low-storage variant – the data structure can use only (near)linear storage, goal: query time as small as possible

Theorem (Agarwal, Matoušek, Sharir '13; Matoušek, **ZP** '15)

Let d, D, s, and $\varepsilon > 0$ be constants. Then the $\Gamma_{d,D,s}$-range searching problem for an arbitrary n-point set in $\mathbb{R}^d$ can be solved with

- $O(n)$ **storage**
- $O(n^{1+\varepsilon})$ *expected* **preprocessing time**
- $O(n^{1-1/d} \log^B n)$ **query time**, *where B is a constant depending on d, D, s and ε.*

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Ramsey thm:

- $\forall k, n$ there exists M s. t. if $|X| = M$ and the set $\binom{X}{k}$ is colored by two colors, there exists an n-element subset of X s.t. all of its k-tuples have the same color.
- $R_k(n)$ = the smallest M with this property

Well-known:

- $R_2(n) = 2^{\Theta(n)}$
- for $k \geq 3$, $\quad \text{twr}_{k-1}(\Omega(n^2)) \leq R_k(n) \leq \text{twr}_k(O(n))$,

  where $\text{twr}_1(x) = x$ and $\text{twr}_{i+1}(x) = 2^{\text{twr}_i(x)}$.

Erdős-Szekerés'35:

- every sufficiently long sequence $(x_1, \ldots, x_M)$ of real numbers contains a subsequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$, $i_1 < \cdots < i_n$, that is either increasing, or nonincreasing.
- Ramsey: $k = 2 \ldots\ldots M \le R_2(n) = 2^{\Theta(n)}$
- but $M = (n-1)^2 + 1$ suffices

coloring is "algebraically defined" – given by a semialgebraic predicate

Semialgebraic setting

- $x_1, \ldots, x_k$ – points in $\mathbb{R}^d$
- $x_{i,j} = j$th coordinate of x_i
- k-ary d-dim semialgebraic predicate $\Phi(x_1, \ldots, x_k)$ – Boolean combination of polynomial equations and inequalities in $x_{i,j}$'s

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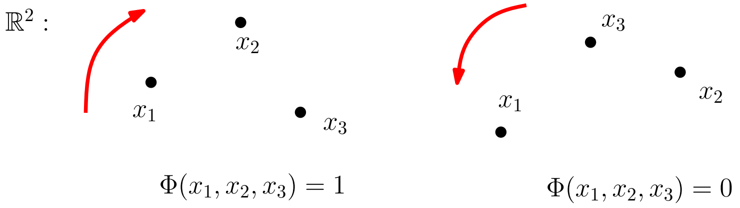
Example: ORDER TYPE

- k-tuples of points in general position having a **clockwise orientation**



- point sequence $(p_1, \ldots, p_n)$ in $\mathbb{R}^d$ is Φ-homogeneous if:
  - $\Phi(p_{i_1}, \ldots, p_{i_k})$ holds for every choice $1 \leq i_1 < \cdots < i_k \leq n$,
  - or it does not hold for any such choice

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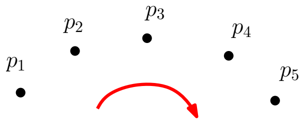
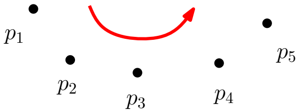
Example: **order-type homogeneous** point sequence

$p_1, \ldots, p_5$ points in general position in $\mathbb{R}^2$:





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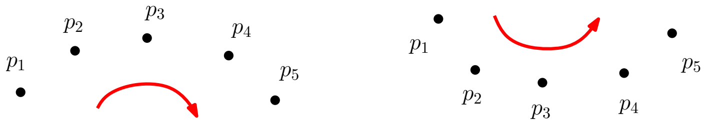

- $R_\Phi(n)$ = the smallest M s. t. every point sequence of length M contains a **Φ-homogeneous subsequence** of length n

- For each k-ary d-dim Φ: $R_\Phi(n) \le \mathrm{twr}_{k-1}(n^C)$, where C depends on d, k, Φ

  $k = 2$ .... Alon, Pach, Pinchasi, Radoičić, Sharir '05
  $k \ge 3$ .... Conlon, Fox, Pach, Sudakov, Suk '14

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- for $k \leq 4$ is the height of the tower $\text{twr}_{k-1}(n^C)$ optimal
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Theorem (Conlon, Fox, Pach, Sudakov, Suk '14)

For every $k \geq 4$ there is a d-dim k-ary semialgebraic predicate Φ s.t. $R_\Phi(n) \geq \mathrm{twr}_{k-1}(\Omega(n))$ where $d = 2^{k-4}$.

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Proof: semialgebraic version of Erdős-Hajnal stepping-up lemma

Application

- $p_1, p_2, p_3, \ldots, p_n$ points in $\mathbb{R}^d$ in general position
- the sequence $(p_1, p_2, \ldots, p_n)$ is order-type homogeneous if all $(d+1)$-tuples $(p_{i_1}, \ldots, p_{i_{d+1}})$, $i_1 < \cdots < i_{d+1}$, have the same sign
- $\mathrm{OT}_d(n) =$ the smallest M s.t. every sequence of length M contains an **order-type homog. subsequence** of length n

Ramsey-type theorem with large Ramsey function

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- $\mathrm{OT}_d(n) \ge \mathrm{twr}_d(\Omega(n))$ ... EM**P**R + Bárány, Matoušek, Pór'13

$\Rightarrow$ **The bound is essentialy tight!**

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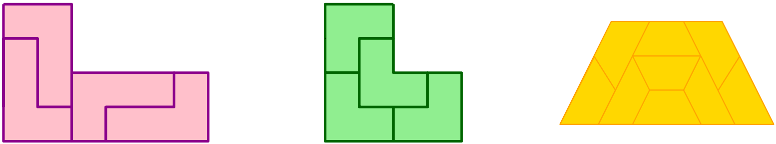
Rep-tiles, or Replicating Tiles, are tiles that can be joined together to make larger replicas of themselves.



Definition

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Problem (Adler, Edmonds, Matoušek)

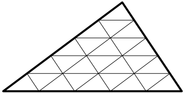
- For what k and d there exist d-dim simplices that are k-reptiles?
- Fix d. What is the smallest k such that there exists a k-reptile?

Motivated by the problem of probabilistic packet marking

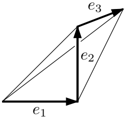
Known results

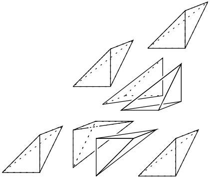
$\mathbb{R}^2$: complete characterization ...... Snover, Waiveris, Williams'91

For example:
every triangle is a k^2-reptile:



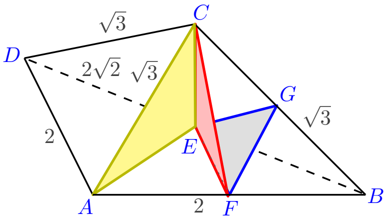
$\Rightarrow$





$\mathbb{R}^d$: Hill simplices are k^d-reptiles

- Hill simplices are not the only d-dim reptiles . . . . . Maehara'13



- For $d \geq 3$ no d-dim simplex is a 2-reptile . . . . Matoušek'05

Theorem (Matoušek, **ZP**, '11)

A tetrahedron can be a k-reptile only for $k = m^3$ for $m = 2, 3, \ldots$

Conjecture (Matoušek, **ZP**, '11)

A d-simplex can be a k-reptile only for $k = m^d$, $m \geq 2, d > 3$.

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Simplified proof of 3-dim case and

Theorem (Kynčl, **ZP**, '15)

A 4-dim. simplex can be a k-reptile only for $k = m^2$ for $m \geq 2$.

Theorem (Matoušek, **ZP**, '11)

A tetrahedron can be a k*-reptile only for* $k = m^3$ *for* $m = 2, 3, \ldots$

Conjecture (Matoušek, **ZP**, '11)

A d-simplex can be a k-reptile only for $k = m^d$, $m \geq 2, d > 3$.

Simplified proof of 3-dim case and

Theorem (Kynčl, **ZP**, '15)

A 4-dim. simplex can be a k*-reptile only for* $k = m^2$ *for* $m \geq 2$.

Still open: Is there a 4-reptile 4-dim simplex?

Thesis available at:

http://kam.mff.cuni.cz/~zuzka/thesis/thesis.pdf

Thank you for your kind attention!