Report on the thesis,
"Non-standard representations of numbers" by Daniel Dombek

Subject of the thesis

This thesis deals with two topics, the negative base β-expansions of real numbers and the unit sum number problem, both of which concern the problem of representing numbers.

β-expansions of real numbers has its origin in the ergodic theory. After Renyi and Parry initiated the study of the β-expansion, it has been studied extensively. Now the β-expansions appear in many fields, ranging from the purely mathematical ones to the engineering applications. This thesis studies a variant of the β-expansions, the $(-\beta)$-expansions which use negative bases $-\beta < -1$. The $(-\beta)$-expansions arise from the orbit of the $(-\beta)$-transformation $T_{-\beta}$ defined by

$$T_{-\beta} : [l, l+1) \ni x \mapsto -\beta x - \lfloor -\beta x - l \rfloor \in [l, l+1),$$

where l is a real number. Ito and Sadahiro first introduced the $(-\beta)$-transformation as the special one with $l = -\beta/(\beta+1)$, and this thesis studies a generalization. After Ito and Sadahiro introduced the $(-\beta)$-expansion, it has been extensively studied by quite a few researchers. The thesis studies $(-\beta)$-expansions mainly from the number theoretical and combinatorial point of view. The main object is the properties of the so-called $(-\beta)$-integers $\mathbb{Z}_{-\beta}$, which is the set of the real numbers having $(-\beta)$-expansions without non-zero fractional parts. It can be said that the thesis pursuits the analogies of the $(-\beta)$-expansion to the ordinal β-expansions, which is never trivial and interesting.

The unit sum number problem is the problem of determining the number of unit required to represent (algebraic) integers in a given field as a sum of units. One of the remarkable results known in this field is the theorem by Jarden and Narkiewicz: For any algebraic number field K, there is no bound N, such that we can express every integer in K as a sum of less than N units of K. The thesis shows generalizations of this theorem.

The thesis also studies a related problem, the distinct unit generated field problem. This problem asks whether every integer in a given field can be represented as a sum of distinct sum.

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Results of the thesis

Part I Part I of the thesis gives preliminaries which is required in the main parts, Part II and Part III.

Part II Chapter 4 introduces the β-expansions and $(-\beta)$-expansions and gives some known results.

Chapter 5 introduces a generalization of $(-\beta)$-expansion, which is one of the main subject of the thesis. It is shown that the expansion $d_{-\beta}(l)$ of the left end point l of the interval on which the $T_{-\beta}$ is defined characterizes the sequence of the coefficients of the $(-\beta)$-expansions.

Chapter 6 is the main chapter of the thesis. It gives some fundamental properties of the $(-\beta)$-integers. Theorem 6.8 and 6.11 gives a explicit formula of the gaps between consecutive two $(-\beta)$-integers. Section 6.2 gives a characterization of $(-\beta)$-integers using *antimorphisms*. In Theorem 6.15, it is shown that the bi-infinite gap sequence of the $(-\beta)$-integers is the fixed point of a certain antimorphism. The antimorphism can be explicitly presented in the case with $l = -\beta/(\beta+1)$. Section 6.3 determines the β for which the spectrum $X(-\beta)$ coincides with the $(-\beta)$-integers $\mathbb{Z}_{-\beta}$. The spectrum $X(-\beta)$ is defined by

$$X(-\beta) = \left\{ \sum_{i=0}^{n} a_i(-\beta)^i \,\middle|\, a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}, n \geq 0 \in \mathbb{Z} \right\}.$$

The antimorphism plays a key role. Section 6.4 shows concrete computational results on many quadratic and cubic β's.

Part III

Part III studies the so-called unit sum number problems and its generalization.

Section 7.2 gives a generalization of a theorem by Jarden and Narkiewicz: Let m be a positive integer. There is no bound N for any algebraic number field K, such that we can express every integer in K as a sum of less than N integers with norms less than m.

Chapter 8 studies the Distinct unit generated field problem, in which the author develops a new methods to extend the list of totally complex quartic fields which are DUG.

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Opinion The thesis contains many interesting and original results, which is written in very clear style. For the proof, the author develops new ideas and techniques. Some of the proofs are quite involved and I did not check all the details, but I am confident that they are correct.

In my opinion, as a whole, this is a good thesis fulfilling the standard requirements for the award of a doctoral degree.

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