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Department of Applied Mathematics

Combinatorial Games Theory

Abstract of the Doctoral Thesis
4I4 - Discrete Models and Algorithms
Tomáš Valla

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S disertační prací je možné se seznámit na studijním oddělení – doktorské studium, MFF UK, Ke Karlovu 3, 120 00, Praha 2.

Contents

1 Introduction

There are many aspects of mathematics, but only few areas are undergoing such rapid growth as the game theory. Also, very few areas of combinatorics display such a variety of applicability to various other parts of mathematics and practical problems.

Since the dawn of humanity, people were attracted to various games, battling themselves in Go, Chess, Backgammon, and countless more games. However, not only these free-time puzzles were called a game. Many important personalities called their actions in the field of for example military, politics and economics to be a "game". All these activities share the following: there are several players (warriors, politicians, businessmen, animal species, ...) trying to maximise their own profit (money, parliament seats, territory gain, ...) in some sense, while on the other hand usually having to cleverly counteract the actions of their opponents. It seems only natural to study, what should be the rational behaviour of all players, given their initial assumptions and situation.

When we are talking about a game, or game setting of certain process, this means we are in the world where there are selfish and greedy entities, which tend to compete each other. It is much harder to control and manoeuvre this environment, when compared to a setting when there is a central authority which may impose any demand on the environment and all entities. Among such "anarchy" setting are for example computer networks on the Internet (hardly anyone can dictate how should the connected computers behave, but still we would like to impose certain protocols and recommended behaviour), democratic economies (we need to preserve some freedom of subjects on the market, but we would still like to collect taxes, punish unfair traders and manipulate the whole economics in a certain direction), and many other examples. Note, that reaching some desired global objective in the "anarchy" setting is therefore much more complicated, at least in general. The authority in the game setting does not have the right to directly control the actions of the players, the only stimulus the authority can apply is the setting of game rules and payoffs of the players according to their behaviour.

And let us not restrict ourselves only to settings with vast number of players/entities. For example, in Ramsey theory, which is a well-known and well-studied part of modern mathematics, we are interested in the smallest size of some combinatorial object (graph, hypergraph, arithmetic progression, ...) that is sufficient for existence of certain internal homogeneity of the object. Let us consider the game setting of this—there are two players, the first one is trying to build the internal homogeneity of the object and the second is trying to build as chaotic object as possible. We may now ask: What is the smallest size of the object such that the first player may always succeed in building the internal homogeneity? This game size should be of course smaller than the size in the Ramsey setting. In fact, in many cases the difference between both sizes is dramatic.

Therefore, based on the previous thoughts, we formulate the general topic of this thesis: *We study the complexity that often suddenly appears, when we consider the competitive version of a certain environment, process or behaviour.*

2 Obtained Results

In accordance with our general topic we present our results from three fields of the game theory: algorithmic game theory, cops and robber games, and positional games.

In the field of algorithmic game theory (Section 3), we prove a positive and somewhat surprising result—when we consider the competitive setting of the vertex cover problem and its generalisations to set cover and some other variants, where each vertex is controlled by an independent player, we show a game that reaches the same global quality vertex (set) cover as in the central-authority setting. Here we have to measure the quality of the reached cover in the terms of approximation of the optimal solution, as the vertex cover is a classical NP-complete problem.

In the field of the cops and robber games (Section 4), we solve an open question regarding the computational complexity of the so-called guarding game problem. This is a negative result, as we show that the problem is E-complete for general graphs, where $\mathrm{E} = \mathrm{DTIME}(2^{\mathcal{O}(n)})$. And finally in the field of positional games (Section 5), we again prove several positive results. We establish the game size of game versions of several Ramsey-type theorems, and we show that the game sizes are much smaller than the Ramsey sizes.

See the end of this report for the list of published or submitted papers on which the thesis is based on.

3 Algorithmic Game Theory: Covering Games

Combinatorial optimisation has for several decades dictated the landscape of algorithm design. One limiting assumption of the "classical" combinatorial optimisation is the existence of an omnipotent centralised authority that has access to all the relevant information and has the power to enforce any solution of its choice. Over the last decade, the soundness of such assumptions has increasingly come into question following a number of paradigm-shifting socioeconomic events such as the rapid rise of the Internet, the painful realization of the extent of inter-connectivity of the global economy as well as the emergence of global sustainability concerns.

The competition between individual incentives and social optimality is of fundamental concern in distributed systems as it can lead to highly inefficient outcomes. The price of anarchy examines exactly what are the worst case repercussions of such a policy. Formally, price of anarchy is defined as the maximal ratio between the social cost of a Nash equilibrium and that of the global optimal configuration. Intuitively, a low price of anarchy implies that upon converging to a socially stable outcome, the quality of the acquired solution is almost optimal from a central optimisation perspective.

Unfortunately, in many cases of interesting games the price of anarchy is extremely high. Vertex cover, due to its prominent position within combinatorial optimisation, has been studied in the context of game theory from different approaches, all of which so far have shared this limiting characteristic.

We present a new class of vertex cover and set cover games, called *Mafia games.* Using this new techniques, we are able to prove the following results.

Theorem 3.1. *The Mafia games for vertex cover, hitting set and submodular hitting set always have pure Nash equilibria, and the price of anarchy is 2 for vertex cover and d for (submodular) hitting set (where d is the size of the largest set in the hitting set problem).*

Thus, in our work, the price of anarchy bounds match the best known constant factor approximation guarantees for the centralised optimisation problems for linear and also for submodular costs—in contrast to all previously studied covering games, where the price of anarchy cannot be bounded by a constant. The rules of the games capture the structure of the linear programming relaxations of the underlying optimisation problems, and our bounds are established by analysing these relaxations. Furthermore, for linear costs we exhibit linear time best response dynamics that converge to these almost optimal Nash equilibria. These dynamics mimic the classical greedy approximation algorithm of Bar-Yehuda and Even.

In the description, we use the colourful and yet intuitive terminology of a Mafia (service points) which "provides security" (covers edges). The vertices may choose to join Mafia or to remain civilians. Each edge of the graph has to be "secured", that is, at least one endpoint must be in Mafia. For player v, there is an initial cost $c(v)$ to join Mafia. Mafiosi can collect ransoms as the price of security of the incident edges: if a vertex v chooses to be a mafioso, his strategy also includes a ransom vector, so that the total ransom he demands from his neighbours is $c(v)$. It is a one-shot game and mafiosi can ransom both their civilian and mafioso neighbours.

If v is a civilian, he has to pay to his neighbours in the Mafia all ransom they demand. Furthermore, if there is an incident uncovered edge uv, that is, u is also a civilian, both of them have to pay a huge penalty. In contrast, if v is a mafioso, he has to pay $c(v)$ for joining, and he receives whatever he can collect from ransoms. However, mafiosi ransomed excessively obtain a protected status: if the total demand from v is more than $c(v)$, he satisfies only a proportional fraction of the demands. It is important to note that the payoff function is defined locally:

besides his own strategy, the payoff of a player depends only on the strategies of players at distance at most 2 from him (i.e. immediate neighbours and neighbours of neighbours). Also note that if M is a vertex cover, then the total utility of the players is $-c(M)$. Consequently, an optimal solution to the optimisation problem gives a social optimum of the game.

Our approach avoids bad Nash equilibria that are possible in Cardinal and Hoefer [11] and Balcan et al. [3]. As an example, consider the vertex cover game on a star with all vertices having cost 1. In the models of [11] and [3] there exists a Nash equilibrium where the leaves form the vertex cover. In our model, if all leaves are mafiosi, then all of them would demand ransom from the central player, who would then have a strong incentive to join the mafia and obtain the protected status. It can be verified that the only Nash equilibria correspond to outcomes where the central vertex and at most one leaf are in the Mafia.

As a different interpretation of the game above, consider a road network with the vertices representing cities. The maintenance of a road must be provided by a facility at one of the endpoints. The cost of opening the facility dominates the operating cost: if city v decides to open one at cost $c(v)$, it is able to maintain all incident roads. As a compensation, the cities can try to recollect the opening cost by asking contributions from the neighbouring cities. A city without a facility has to pay all contributions he is asked to pay. However, if a city opens a facility, its liability is limited and has to satisfy demands only up to his opening cost, $c(v)$.

Our approach can be extended to the hitting set problem, which is equivalent to the set cover problem. We are given a hypergraph $G = (V, \mathcal{E})$, and a cost function $c\, V \to \mathcal{R}_+$ on the vertices. Our aim is to find a minimum cost subset M of V intersecting every hyperedge in $\mathcal{E}$. This problem is known to be approximable within a factor of d, the maximum size of a hyperedge. In the corresponding Mafia game, the hyperedges shall be considered as clubs in need of security. A mafioso can assign ransoms to the clubs he is a member of, that will be distributed equally to all other members of the club.

We shall prove that for the vertex cover and hitting set games, the price of anarchy is 2 and d, respectively. Bar-Yehuda and Even gave a simple primal-dual algorithm with this guarantee in 1981 [4]. No better constant factor approximation has been given ever-since. Furthermore, assuming the Unique Games Conjecture, Khot and Regev [28] proved that the hitting set problem cannot be approximated by any constant factor smaller than d.

As a further extension, we also investigate the submodular hitting set (or set cover) problem, that has received significant attention recently. The goal is to find a hitting set M of a hypergraph minimising $C(M)$ for a submodular set function C on the ground set. Independently, Koufogiannakis and Young [30] and Iwata and Nagano [26] gave d-approximation algorithms. Our game approach extends even to this setting, with the same price of anarchy d. This involves a new player, the Godfather, who's strategy consists of setting a budget vector in the submodular base polyhedron of C. Otherwise, the game is essentially the same as the (linear) hitting set game.

Recent work of Roughgarden et al. [43], [8], [44] has shown that the majority of positive results in price of anarchy literature can be reduced to a specific common set of structural assumptions. In contrast, in our work, we use a novel approach by exploring connections to the LP relaxations of the underlying centralised optimisation problems. This connection raises interesting questions about the limits of its applicability.

The world of decentralised competition is not immune to the results of computational complexity. Hence, a low price of anarchy does not necessarily yield a usable outcome in the means of the game dynamics, when players sequentially have the possibility to change their strategies for a better one.

We prove the following.

Theorem 3.2. *For the vertex cover and hitting set covering games, there is a dynamics such that after $\mathcal{O}(n)$ moves, we obtain a strategy profile in Nash equilibrium.*

In our covering games, we first show that even in simple instances, round robin best response dynamics may end in a loop. However, this can be simply fixed by a slight modification of the payoff.

We introduce a secondary utility, that does not affect the price of anarchy results, but merely instigates the mafiosi to use more fair (symmetric) ransoms: $r(u, v) = r(v, u)$. With this secondary objective, we show that actually a single round of best response dynamics under a simple selection rule of the next player results in a Nash-equilibrium. This dynamics in fact simulates the Bar-Yehuda–Even algorithm. An analogous dynamics is shown in the case of hitting set. Moreover, these dynamics can be interpreted in a distributed manner, enabling several players to change their strategies at the same time.

Our results were distantly inspired by the approach of Panagopoulou and Spirakis [41], who described a vertex colouring game and analysed its price of anarchy. We thus looked for a different important combinatorial problem with a potential to be analysed in the competitive setting. There were also previous attempts to define and analyse covering games, but unfortunaly these had a high price of anarchy.

3.1 Vertex cover game definition

Given a graph $G = (V, E)$, let $c\,V \to \mathcal{R}^+$ be a cost function on the vertices. In the *vertex cover problem*, the task is to find a minimum cost set $M \subseteq V$ containing at least one endpoint of every edge in E. For a vertex $v \in V$, let $N(v) = \{u;\ \{u, v\} \in E\}$ denote the set of its neighbours.

The *Mafia Vertex Cover Game* is a one-shot game on the player set V. The basic strategy of an player is to decide being a civilian or a mafioso. The set of civilians shall be denoted by C, the set of mafiosi (Mafia) by M. For civilians, no further decision has to be made, while for mafiosi, their strategy also contains a ransom vector. Each mafioso $m \in M$ can demand ransoms from his neighbours totalling $c(m)$. The ransom demanded from a neighbour $u \in N(m)$ is $r(m,u) \geq 0$, with $\sum_{u \in N(v)} r(m,u) = c(m)$. The strategy profile $\mathcal{S} = (M, C, r)$ thus consists of the sets of mafiosi and civilians, and the ransom vectors.

Let us call $c(v)$ the *budget* of an player $v \in V$, and let $T > \sum_{v \in V} c(v)$ be a huge constant. Let

$$D(v) = \sum_{m \in M} r(m,v)$$

be the demand asked from the player $v \in V$.

Let us now define the payoffs for a given strategy profile $\mathcal{S}$. For a civilian $v \in C$, let $\mathrm{P}en(v) = T$ if v is incident to an uncovered edge, that is $C \cap N(v) \neq \emptyset$, and $\mathrm{P}en(v) = 0$ otherwise. The utility of $v \in C$ is

$$u_{\mathcal{S}}(v) = -D(v) - \mathrm{P}en(v).$$

If $v \in M$ and the total demand from v is $D(v) > c(v)$ (i.e. v is asked too much), we call v *protected* and denote the set of protected mafiosi by $P \subseteq M$. The real amount of money that the protected mafioso $p \in P$ pays to his neighbours is scaled down to $\frac{c(p)}{D(p)} r(u,p)$. Let $F^{-}(v) = \min\{D(v), c(v)\}$ be the total amount the mafioso v pays for ransom. Let

$$F^{+}(v) = \sum_{u \in N(v) \setminus P} r(v,u) + \sum_{u \in N(v) \cap P} \frac{c(u)}{D(u)} r(v,u)$$

denote the income of $v \in M$ from the ransoms. Then the utility of a mafioso $v \in M$ is defined as

$$u_{\mathcal{S}}(v) = -c(v) + F^{+}(v) - F^{-}(v).$$

This means v has his initial cost $c(v)$ for entering the Mafia, receives full payment from civilians and unprotected mafiosi, receives reduced payment from protected mafiosi, and pays the full payment to his neighbouring mafiosi if v is unprotected, or reduced payment if v is protected.

3.2 The existence of pure Nash equilibria

Pure Nash equilibria are (deterministic) strategy outcomes such that no player can improve her payoff by unilaterally changing her strategy. We will start by establishing that our game always exhibits such states. The following is the standard linear programming relaxation of vertex cover along with its dual.

$$\min \sum_{v \in V} c(v)x(v) \tag{P-VC}$$

$$x(u) + x(v) \geq 1 \quad \text{for all } \{u, v\} \in E \tag{1}$$

$$x \geq 0 \tag{2}$$

$$\max \sum_{\{u,v\} \in E} y(\{u, v\}) \tag{D-VC}$$

$$\sum_{\{u,v\} \in E} y(\{u, v\}) \leq c(u) \quad \text{for all } u \in V \tag{3}$$

$$y \geq 0 \tag{4}$$

For a feasible dual solution y we say that the vertex $v \in V$ is *tight* if

$$\sum_{\{u,v\} \in E} y(\{u, v\}) = c(v).$$

We call the pair (M, y) a *complementary pair* if M is a vertex cover, y is a feasible dual solution, and each $v \in M$ is tight with respect to y.

Lemma 3.3. *If (M, y) is a complementary pair, then M is a 2-approximate solution to the vertex cover problem.*

Lemma 3.4. *Let (M, y) be a complementary pair, and consider the strategy profile where the players in M form the Mafia and $C = V \setminus M$ are the civilians. For $u \in M$, define $r(u, v) = y(\{u, v\})$ for every $v \in N(u)$. Then the strategy profile $\mathcal{S} = (M, C, r)$ is a Nash equilibrium.*

The existence of a complementary pair is provided by the algorithm of Bar-Yehuda and Even [4].

Using Lemma 3.4, we obtain the following theorem.

Theorem 3.5. *The Mafia Vertex Cover Game always has a pure Nash equilibrium.*

3.3 The Price of Anarchy

For a strategy profile $\mathcal{S}$ with a uncovered edges, the sum of the utilities is $-c(M) - 2aT$. The Price of Anarchy compares this sum in a Nash equilibrium at the worst case to the maximum value over all strategy profiles, that corresponds to a minimum cost vertex cover.

Lemma 3.6. *Let the strategy profile $\mathcal{S} = (M, C, r)$ be a Nash equilibrium. Then there are no protected mafiosi.*

Lemma 3.7. *Suppose the strategy profile $\mathcal{S} = (M, C, r)$ is a Nash equilibrium and let $v \in C$. Then $D(v) \leq 2c(v)$.*

Theorem 3.8. *The price of anarchy in the Mafia game is 2.*

3.4 Set cover and hitting set

In this section, we generalise our approach to the hitting set problem. Given a hypergraph $\mathcal{G} = (V, \mathcal{E})$ and a cost function $cV \to \mathcal{R}_+$, we want to find a minimum cost $M \subseteq V$ intersecting every hyperedge. Let $d = \max\{|S|;\ S \in \mathcal{E}\}$.

In the *set cover problem*, we have a ground set U and a collection of subsets $\mathcal{S}$ of U. For a cost function $c\,\mathcal{S} \to \mathcal{R}_+$ we want to find minimum cost collection of subsets whose union is U. This is equivalent to the hitting set problem, where the ground set is $\mathcal{S}$, and to each $u \in U$, there is a corresponding hyperedge that is the collection of subsets containing u.

For simplicity, we define the hitting set game on a d-uniform hypergraph. This can be done without loss of generality. To verify this, take an arbitrary instance $\mathcal{G} = (V, \mathcal{E})$, and let $T > d\sum_{v \in V} c(v)$. Extend V by $d-1$ new vertices of cost T, and for every $S \in \mathcal{E}$, extend S by any $d - |S|$ new elements. If there is a d-approximate solution to the modified instance, it cannot contain any of the new elements. Hence finding a d-approximate solution is equivalent in the original and in the modified instance.

We define the *Mafia Hitting Set Game* on a d-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$. The set of players is V, with $v \in V$ having a *budget* $c(v)$. We shall call the hyperedges *clubs*. For an player $v \in V$, let $\mathcal{N}(v) \subseteq \mathcal{E}$ denote the set of clubs containing v. The players again choose from the strategy of being a civilian or being a mafioso, denoting their sets by C and M, respectively. The strategies of the mafioso m incorporates the ransoms $r(m, S)$ for the clubs S containing m, with $\sum_{S \in \mathcal{N}(v)} r(m, S) = c(m)$.

We define the payoffs for the strategy profile $\mathcal{S} = (M, C, r)$ similarly to the vertex cover case. For a civilian $v \in C$, $\mathrm{P}en(v) = T$ for a large constant T if v participates in a club containing no mafiosi, and 0 otherwise.

In each club S, the ransom $r(m, S)$ of a mafioso $m \in S \cap M$ has to be payed by all other members at equal rate, that is, everyone pays $\frac{r(m,S)}{(d-1)}$ to m. The demand from an player is the total amount he has to pay in all clubs he is a member of, that is,

$$D(v) = \frac{1}{d-1} \sum_{S \in \mathcal{N}(v)} \sum_{m \in M \cap S} r(m, S).$$

Note that if a mafioso v would receive all the money he demands, he would gain

$(d-1)c(v)$. The utility of a civilian $v \in C$ is defined as

$$u_{\mathcal{S}}(v) = -D(v) - \mathrm{P}en(v).$$

A mafioso v receives the protected status if $D(v) > c(v)$, that is, he is asked for more than his maximal income. The set of protected mafiosi is denoted by P, and they pay proportionally reduced ransoms. Let $F^-(v) = \min\{D(v), c(v)\}$ be the total amount v pays. The income is defined by

$$F^+(v) = \sum_{S \in \mathcal{N}(v)} \frac{r(v,S)}{d-1} \left(|S \setminus (P \cup \{v\})| + \sum_{u \in (S \cap P) \setminus \{v\}} \frac{c(u)}{D(u)} \right).$$

The utility of a mafioso $v \in M$ is then

$$u_{\mathcal{S}}(v) = -c(v) + F^+(v) - F^-(v).$$

The standard LP-relaxation extends the formulations (P-VC) and (D-VC).

$$\min \sum_{v \in V} c(v)x(v) \tag{P-HS}$$

$$\sum_{u \in S} x(u) \geq 1 \quad \text{for all } S \in \mathcal{E} \tag{5}$$

$$x \geq 0 \tag{6}$$

$$\max \sum_{S \in \mathcal{E}} y(S) \tag{D-HS}$$

$$\sum_{S \in \mathcal{N}(u)} y(S) \leq c(u) \quad \text{for all } u \in V \tag{7}$$

$$y \geq 0 \tag{8}$$

Again, for a feasible dual solution y, $v \in V$ is called tight if the corresponding inequality in (D-HS) holds with equality. A pair (M, y) of a hitting set M and a feasible dual y is called a *complementary* pair if the dual inequality corresponding to any $v \in M$ is tight.

Lemma 3.9. *If (M, y) is a complementary pair, then M is a d-approximate solution to the hitting set problem.*

Lemma 3.10. *Let us define strategies in the Mafia Hitting Set Game based on a complementary pair (M, y) as follows. Let players in M be the Mafia and $V \setminus M$ be the civilians. For each $v \in M$, define $r(v, S) = y(S)$ for every $S \in \mathcal{E}$ containing v. Then the strategy profile $\mathcal{S} = (M, C, r)$ is a Nash equilibrium.*

Theorem 3.11. *The Mafia Hitting Set Game always has a pure Nash equilibrium.*

Lemma 3.12. *Let the strategy profile $\mathcal{S} = (M, C, r)$ be a Nash equilibrium. Then there are no protected mafiosi.*

Lemma 3.13. *Let the strategy profile $\mathcal{S} = (M, C, r)$ be a Nash equilibrium and let $v \in C$. Then $D(v) \leq \frac{d}{d-1}c(v)$.*

Theorem 3.14. *The price of anarchy for the Mafia Hitting Set Game is d.*

3.5 Submodular hitting set

In the submodular hitting set problem, we are given a hypergraph $G = (V, \mathcal{E})$ with a submodular set function $C\ 2^V \to \mathcal{R}_+$, that is, $C(\emptyset) = 0$, and

$$C(X) + C(Y) \geq C(X \cap Y) + C(X \cup Y) \quad \text{for all } X, Y \subseteq V.$$

We shall assume also that C is monotone, that is, $C(X) \leq C(Y)$ if $X \subseteq Y$. Our aim is to find a hitting set M minimising $C(M)$.

Koufogiannakis and Young [30], and Iwata and Nagano [26] obtained d-approximation algorithms for this problem, where d is the maximum size of a hyperedge.

For a submodular function C, it is natural to define the following two polyhedra. The *submodular polyhedron* is

$$P(C) = \{z \in \mathcal{R}^V;\ z \geq 0, z(Z) \leq C(Z) \text{ for all } Z \subseteq V\},$$

and the *submodular base polyhedron* is

$$B(C) = \{z \in \mathcal{R}^V;\ z \geq 0, z(Z) \leq C(Z) \text{ for all } Z \subsetneq V, z(V) = C(V)\}.$$

Given a vector $z \in P(C)$, the set Z is *tight* with respect to z if $z(Z) = C(Z)$. An elementary consequence of submodularity is that for every $z \in P(C)$, there exists a unique maximal tight set. Note that $B(C) \subseteq P(C)$ and $z \in P(C)$ is in $B(C)$ if and only if V is tight.

Now we define the game. We introduce a new player, the *Godfather* to set the budgets of the players.

The *Submodular Mafia Hitting Set Game* is defined on a hypergraph $\mathcal{G} = (V, \mathcal{E})$ and a monotone submodular set function $C\ 2^V \to \mathcal{R}_+$. There are $|V| + 1$ players, one for each vertex and a special player g, called the *Godfather*.

The strategy of the Godfather is to return a budget vector $\tilde{c} \in B(C)$. The basic strategy of an player $v \in V$ is to decide being a civilian or being a mafioso. The strategy of a mafioso m further incorporates normalised ransoms $r_0(m, S) \geq 0$ for clubs $S \in \mathcal{N}(m)$ with $\sum_{S \in \mathcal{N}(m)} r_0(m, S) = 1$, that is, $r_0(m, S)$ expresses the fraction of the budget of m he is willing to charge on S.

The sets of civilians and mafiosi will again be denoted by C and M, respectively. Hence a strategy profile is given as $\mathcal{S} = (M, C, \tilde{c}, r_0)$. The actual ransoms will be $r(m, S) = r_0(m, S) \cdot \tilde{c}(m)$.

The utility of the Godfather is the total budget of the Mafia: $u_{\mathcal{S}}(g) = C(M)$. The utility of the vertex players is defined the same way as for the linear Mafia Hitting Set Game, with replacing $c(v)$ by $\tilde{c}(v)$ everywhere.

For linear cost functions, we have $C(Z) = \sum_{v \in Z} c(z)$. Then the only vector in $B(C)$ is c, hence the Godfather has only one strategy to choose. Therefore we obtain the same game as the Mafia hitting set game.

Using techniques inpired by the previous cases of vertex cover and hitting set games, we prove the following.

Theorem 3.15. *The price of anarchy for the Submodular Mafia Hitting Set Game is d.*

3.6 Convergence to Nash equilibrium

In this section, we investigate if the Mafia Games defined in the previous section converge under certain best response dynamics. We first show that already in the Mafia Vertex Cover Game, a round robin best response dynamics may run into a loop.

Motivated by this example, we modify the utilities by adding a secondary payoff, that instigates the mafiosi to use symmetric ransoms: $r(u, v) = r(v, u)$. With this secondary objective, we show that a single round of best response dynamics under a simple selection rule results in a Nash-equilibrium. This dynamics simulates the Bar-Yehuda and Even algorithm. An analogous result is then proved for hitting set.

Let us now show an example where a round robin dynamics does not necessarily converge.

Claim 3.16. *On a star on the vertices v_1, v_2, v_3, v_4 and the central vertex z, the round robin dynamics does not necessarily converge.*

If we could incentivise z to change his strategy less drastically and ransom the other players by at most 1, we could rapidly reach a Nash-equilibrium. To enforce such a behaviour, we introduce a secondary utility function.

For a strategy profile $\mathcal{S} = (M, C, r)$, $u_{\mathcal{S}}(v)$ is the utility as defined for the vertex cover case. Let us define $\tilde{u}_{\mathcal{S}}(v) = 0$ if $v \in C$ and

$$\tilde{u}_{\mathcal{S}}(v) = - \sum_{\{u,v\} \in E, u \in M} |r(u, v) - r(v, u)|$$

if $u \in M$. The total utility is then $(u_{\mathcal{S}}(v), \tilde{u}_{\mathcal{S}}(v))$ in the lexicographic ordering: the players' main objective is to maximise $u_{\mathcal{S}}(v)$, and if that is the same for two outcomes, they choose the one maximising $\tilde{u}_{\mathcal{S}}(v)$. In the above example, the dynamics would reach an equilibrium in the second round, with $r(z, v_i) \leq 1$ for all i.

$\tilde{u}_{\mathcal{S}}(v) \leq 0$ and equality holds if $r(u, v) = r(v, u)$ for every $\{u, v\} \in M$, $u, v \in M$. Therefore all results for the vertex cover case remain valid: in Lemma 3.4 we define a strategy profile where $\tilde{u}_{\mathcal{S}}(v) = 0$ for all players, hence it also gives a Nash equilibrium for the extended definition of utilities. The secondary utility term $\tilde{u}$ does not affect the proofs in for the vertex cover case.

Consider now the following simple dynamics: *Start from the strategy profile where all players are civilians. In each step, take an player who is incident to uncovered edge and subject to this, minimises $c(v) - D(v)$, and give him the opportunity to change his strategy.*

Theorem 3.17. *In the vertex cover mafia game, after each player changing his strategy at most once, we obtain a strategy profile in Nash equilibrium.*

The above dynamics can be naturally interpreted in a distributive manner. In the proof of Theorem 3.17, we only use that the vertex v changing his strategy *is a local minimiser* of $c(v) - D(v)$. The simultaneous move of two players u and v could interfere only if $\{u, v\} \in E$ or they have a neighbour t in common. In this case, $c(t) < D(t)$ could result if both u and v start ransoming t simultaneously.

We assume that the players have a hierarchical ordering $\prec$: $u \prec v$ expresses that v is more powerful than u. We call an player v *a local minimiser* if $v \in C$, v is incident to some uncovered edges, and $c(v) - D(v) \leq c(u) - D(u)$ whenever $u \in C$, $\{u, v\} \in E$. A local minimiser v is then called *eligible* if $u \prec v$ for all local minimisers u whose distance from v is at most 2.

We start from $C = V$. In each iteration of the dynamics, we let all eligible players change their strategy to a best response simultaneously. As in the proof of Theorem 3.17, $c(v) - D(v) \geq 0$ is maintained for all $v \in V$, and thus the dynamics terminates after each player changes his strategy at most once.

There are multiple distributed algorithms in the literature for vertex cover, e.g. [29], [23], [31]. The distributed algorithm by Koufogiannakis and Young [31] computes in $\mathcal{O}(\log n)$ rounds a 2-approximation in expectation with high probability. In contrast, we cannot give good bounds on the number of iterations of our distributed dynamics. For example, if the graph is a path $v_1 \dots v_n$, and the budgets are $c(v_i) = i$, then only player i will move in step i. Yet we believe that our dynamics could be practically efficient.

The natural generalisation of the secondary objective for hitting set is as follows. For a club $S \in \mathcal{E}$, let (S) denote the maximum difference between ransoms on this edge. That is, $(S) = 0$ if $|S \cap M| \leq 1$ and $(S) = \max_{v \in S \cap M} r(v, S) - \min_{v \in S \cap M} r(v, S)$ otherwise. For a strategy profile $\mathcal{S} = (M, C, r)$, let $\tilde{u}_{\mathcal{S}}(v) = 0$ if $v \in C$ and

$$\tilde{u}_{\mathcal{S}}(v) = - \sum_{v \in \mathcal{N}(v)} (S)$$

if $v \in M$. The utility of an player is then $(u_{\mathcal{S}}(v), \tilde{u}_{\mathcal{S}}(v))$, under lexicographic ordering.

A natural expectation would be to prove rapid convergence as for vertex cover, if always the player minimising $c(v) - D(v)$ is allowed to play. However, the Bar-Yehuda and Even algorithm does not seem to be modelled by this dynamics. Instead, we define a slightly different extremal choice of the next player. Let

$$D^*(v) = \sum_{S \in \mathcal{N}(v)} \max_{m \in S \cap M} r(m, S),$$

that is, for each club S we consider the largest ransom demanded in this club. Note that $D(v) \leq D^*(v)$. Let us consider the following dynamics. *We start from the strategy profile where everyone is civilian, and we always let a civilian play next who is contained in an uncovered club. Among them, we let the one play who minimises $c(v) - D^*(v)$.*

Theorem 3.18. *In the hitting set mafia game, after each player changing his strategy at most once, we obtain a strategy profile in Nash equilibrium.*

One would expect that the Submodular Mafia Hitting Set Game also converges under some dynamics that simulates the primal-dual algorithm by Iwata and Nagano [26]. However, if the Godfather does not have a secondary utility, the exists an example where it can run into a loop.

We conjecture that with this secondary utility and the secondary utilities for the vertex players as for hitting set, rapid convergence can be shown under an appropriate choice of the next player.

3.7 Conclusions and further research

An intriguing question is if a similar game theoretic approach could be applied for further combinatorial optimisation problems.

The first natural direction would be to extend our approach to a broader class of covering games. The most general approximation result on covering games is [30], giving a d-approximation algorithm for minimising a submodular function under monotone constraints, each constraint dependent on at most d variables. As a first step, one could study hitting set with the requirement that each hyperedge S must be covered by at least $h(S) \geq 1$ elements; a simple primal-dual algorithm was given in [25]. However, extending our game even to this setting does not seem straightforward.

One could also try to formulate analogous settings for classical optimisation problems such as facility location, Steiner-tree or knapsack. One inherent difficulty is that in our analysis, it seems to be crucial that any greedily chosen maximal feasible dual solution gives a good approximation. Also, we heavily rely on the fact that each constraint contains at most d variables.

4 Cops & Robber Games: The Guarding Game

The guarding game is a game in which several cops try to guard a region in a (directed or undirected) graph against a robber. The robber and the cops are placed on the vertices of the graph; they take turns in moving to adjacent vertices (or staying), cops inside the guarded region, the robber on the remaining vertices (the robber-region). The goal of the robber is to enter the guarded region at a vertex with no cop on it. The problem is to determine whether for a given graph and given number of cops the cops are able to prevent the robber from entering the guarded region. Fomin et al. [15] proved that the problem is NP-complete when the robber-region is restricted to a tree. Further they prove that is it PSPACE-complete when the robber-region is restricted to a directed acyclic graph, and they ask about the problem complexity for arbitrary graphs.

4.1 Introduction and motivation

The *guarding game* (G, V_C, c), introduced by Fomin et al. [15], is played on a graph $G = (V, E)$ (or directed graph $G = (V, E)$) by two players, the *cop-player* and the *robber-player*, each having his pawns (c cops and one robber, respectively) on V. There is a protected region (also called cop-region) $V_C \subset V$. The remaining region $V \setminus V_C$ is called robber-region and denoted V_R. The robber aims to enter V_C by a move to a vertex of V_C with no cop on it. The cops try to prevent this. The game is played in alternating turns. In the first turn the robber-player places the robber on some vertex of V_R. In the second turn the cop-player places his c cops on vertices of V_C (more cops can share one vertex). In each subsequent turn the respective player can move each of his pawns to a neighbouring vertex of the pawn's position (or leave it where it is). However, the cops can move only inside V_C and the robber can move only on vertices with no cops. At any time of the game both players know the positions of all pawns. The robber-player wins if he is able to move the robber to some vertex of V_C in a finite number of steps. The cop-player wins if the cop-player can prevent the robber-player from placing the robber on a vertex in V_C indefinitely. Note that after exponentially many (in the size of the graph G) turns the positions has to repeat and obviously if the robber can win, he can win in less than $2|V|^{(c+1)}$ turns, Note that $2|V|^{c+1}$ is the upper bound on the number of all possible positions of the robber and all cops, so after that many turns the position has to repeat. Thus, if the robber can win, he can win in less than $2|V|^{c+1}$ turns. Consequently, we may define the robber to lose if he does not win in $2|V|^{c+1}$ turns.

For a given graph G and guarded region V_C, the task is to find the minimum number c such that cop-player wins. Note that this problem is polynomial-time equivalent with the problem of determining the outcome of the game for a fixed number c of cops.

The guarding game is a member of a big class called the pursuit-evasion games, see, e.g., Alspach [2] for introduction and survey. The discrete version of pursuit-evasion games on graphs is called the Cops-and-Robber game. This game was first defined for one cop by Winkler and Nowakowski [40] and by Quilliot [42]. Aigner and Fromme [1] initiated the study of the problem with several cops. The minimum number of cops required to capture the robber is called the cop number of the graph. In this setting, the Cops-and-Robber game can be viewed as a special case of search games played on graphs. Therefore, the guarding game is a natural variant of the original Cops-and-Robber game. The complexity of the decision problem related to the Cops-and-Robbers game was studied by Goldstein and Reingold [20]. They have shown that if the number of cops is not fixed and if either the graph is directed or initial positions are given, then the problem is E-complete. Another interesting variant is the "fast robber" game, which is studied in Fomin et al. [17]. See the annotated bibliography [19] for reference on further topics.

A different well-studied problem, the *Eternal Domination* problem (also known as *Eternal Security*) is strongly related to the guarding game. The objective in the Eternal Domination is to place the minimum number of guards on the vertices of a graph G such that the guards can protect the vertices of G from an infinite sequence of attacks. In response to an attack of an unguarded vertex v, at least one guard must move to v and the other guards can either stay put, or move to adjacent vertices. The Eternal Domination problem is a special case of the guarding game. This can be seen as follows. Let G be a graph on n vertices and we construct a graph H from G by adding a clique K_n on n vertices and connecting the clique and G by n edges which form a perfect matching. The cop-region is $V(G)$ and the robber-region is $V(K_n)$. Now G has an eternal dominating set of size k if and only if k cops can guard $V(G)$.

We focus on the complexity issues of the following decision problem: Given the guarding game $\mathcal{G} = (G, V_C, c)$, who has the winning strategy?

Let us define the computational problem precisely. The *directed guarding decision problem* is, given a guarding game (G, V_C, c) where G is a directed graph, to decide whether it is a cop-win game or a robber-win game. Analogously, we define the *undirected guarding decision problem* with the difference that the underlying graph G is undirected. The *guarding problem* is, given a directed or undirected graph G and a cop-region $V_C \subseteq V(G)$, to compute the minimum number c such that the (G, V_C, c) is a cop-win.

The directed guarding decision problem was introduced and studied by Fomin et al. [15]. The computational complexity of the problem depends heavily on the chosen restrictions on the graph G. In particular, in [15] the authors show that if the robber's region is only a path, then the problem can be solved in polynomial time, and when the robber moves in a tree (or even in a star), then the problem is NP-complete. Furthermore, if the robber is moving in a directed acyclic graph, the problem becomes PSPACE-complete. Later Fomin, Golovach and Lokshtanov [18] studied the *reverse guarding game* with the same rules as in the guarding game, except that the cop-player plays first. They proved in [18] that the related decision problem is PSPACE-hard on undirected graphs. Nagamochi [32] has also shown that that the problem is NP-complete even if V_R induces a 3-star and that the problem is polynomial-time solvable if V_R induces a cycle. Also, Thirumala Reddy, Sai Krishna and Pandu Rangan have proved [46] that if the robber-region is an arbitrary undirected graph, then the decision problem is PSPACE-hard.

Fomin et al. [15] asked the following question.

Question 4.1. *(Fomin et al.) Is the guarding decision problem for general graphs PSPACE-complete?*

4.2 The complexity of the problem

Let us consider the class E $=$ DTIME($2^{\mathcal{O}(n)}$) of languages recognisable by a deterministic Turing machine in time $2^{\mathcal{O}(n)}$. We consider log-space reductions, this means that the reducing Turing machine is log-space bounded. In pursuit of Question 4.1 we prove the following result.

Theorem 4.1. *The directed guarding decision problem is E-complete under log-space reductions.*

We would like to point out the fact that we can prove Theorem 4.1 without prescribing the starting positions of players. Immediately, we get the following corollary.

Corollary 4.2. *The guarding problem is E-complete under log-space reductions.*

Let us explain here the relevance of Theorem 4.1 to Question 4.1. Very little is known how the class E is related to PSPACE. It is only known [9] that E $\neq$ PSPACE. The following corollary shows that positive answer to Question 4.1 would give a relation between these two complexity classes. This gives unexpected and strong incentive to find positive answer to Question 4.1. (On the other hand, to the skeptics among us, it may also indicate that negative answer is more likely.)

Corollary 4.3. *If the conjecture of Fomin et al. is true, then $E \subseteq PSPACE$.*

We also prove Theorem 4.4, a theorem similar to Theorem 4.1 for general undirected graphs. We define the *guarding game with prescribed starting positions* $\mathcal{G} = (G, V_C, c, S, r)$, where $S : \{1, \ldots, c\} \to V_C$ is the initial placement of cops and $r \in V_R$ is the initial placement of robber. The *undirected guarding decision problem with prescribed starting positions* is, given a guarding game with prescribed starting positions (G, V_C, c, S, r) where G is an undirected graph, to decide whether it is a cop-win game or a robber-win game. The *directed guarding decision problem with prescribed starting positions* is defined analogously.

Theorem 4.4. *The undirected guarding decision problem with prescribed starting positions is E-complete under log-space reductions.*

Here, we would like to point out the fact that with the exception of the result in [18], all known hardness results for cops and robbers, or pursuit evasion games are for the directed graph variants of the games [15], [20]. For example, the classical Cop and Robbers game was shown to be PSPACE-hard on directed graphs by Goldstein and Reingold in 1995 [20] while for *undirected* graphs, even an NP-hardness result was not known until recently by Fomin, Golovach and Kratochvíl [16].

Let us also consider the guarding game $\mathcal{G}^R = (G, V_C, c)^R$, where the two initial turns are different: In the first turn, the cop-player places all cops on vertices of V_C, and in the second turn, the robber-player places the robber on some vertex of V_R. Then the game proceeds as usual, starting with the cop-player. In some sense, this game looks like being harder for the cop-player, because the robber during his initial placement has better chance to endanger the cop-region. Analogously with the definition of directed guarding decision problem we define the *reverse directed guarding decision problem*.

Fomin, Golovach and Lokshtanov [18] proved that the reverse undirected guarding problem is PSPACE-hard. We show the following theorem for the directed case.

Theorem 4.5. *The reverse directed guarding decision problem is E-complete under log-space reductions.*

For the original Cops-and-Robber game, Goldstein and Reingold [20] have proved that if the number c of cops is not fixed and if either the graph is directed or initial positions are given, then the related decision problem is E-complete.

In a sense, we show analogous result for the guarding game as Goldstein and Reingold [20] have shown for the original Cops-and-Robber game. Similarly to Goldstein and Reingold, we can prove the complexity of the undirected guarding decision problem only when having prescribed the initial positions of players. Dealing with this issue seems to be a nontrivial task in this family of games.

4.3 Further Questions

As we have already mentioned, the relation of the classes PSPACE and E is unclear as we only know that PSPACE $\neq$ E and the current state of the art is missing some deeper understanding of the relation. Therefore, the conjecture of Fomin et al. whether the guarding problem is PSPACE-complete still remain open. However, we believe that the conjecture is not true.

For a guarding game $\mathcal{G} = (G, V_C, c)$, what happens if we restrict the size of strongly connected components of G? If the sizes are restricted by 1, we get DAG, for which the decision problem is PSPACE-complete. For unrestricted sizes we have shown that $\mathcal{G}$ is E-complete. Is there some threshold for $\mathcal{G}$ to become E-complete from being PSPACE-complete? This may give us some insight into the original conjecture. We are also working on forcing the starting position in the guarding game on undirected graphs in a way similar to Theorem 4.1.

5 Positional Games: Structural Ramsey Games

Given a certain object, Ramsey theory states that there exists an *internal regularity* inside, some homogeneous sub-object. To the contrary, the goal of many combinatorial games is to *create* such a homogeneous sub-objects. Usually, the validity of Ramsey-type theorems depends only on the size of the object; given object large enough, the theorem holds. We call such minimal sufficient size *Ramsey number.* Similar concept exists in combinatorial games. By *game number* we mean the minimum object size such that certain player (usually the first) wins, provided he uses the best strategy possible. Often, there is large gap between the Ramsey number and the appropriate game number.

In this chapter we study various Ramsey-type theorems and the corresponding games, establish good upper bounds on both numbers and discover large gaps between them. In many cases, establishing a reasonable Ramsey number upper bound is an enormously complicated task which has been a subject of effort of many great mathematicians. Surprisingly, when considering the corresponding combinatorial game, it is often quite easy to find good upper bound on the game number, usually much lower than the Ramsey number bound. Therefore, we consider this topic exceptionally interesting. In particular, we contribute to this topic by investigating games corresponding to structural extensions of Ramsey and van der Waerden theorems—Brauer theorem, structural and restricted Ramsey theorems. The results were published in the paper [39].

5.1 Introduction

Ramsey theory deals with the statements of the following type: For every partition $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$ of the set $\binom{C}{A}$ of all substructures of C which are isomorphic to A, there exists a substructure B of C such that the set $\binom{B}{A}$ belongs to one class of the partition. This definition of course assumes that we make precise notions of the structure and of the substructure. The validity of the previous statement is denoted by $C \to (B)^A_k$. Every $B' \in \binom{C}{B}$ is called a *copy* of B in C.

The classical Ramsey theorem in this setting claims that for all integers k, n, p there exists an integer N such that

$$K_N \to (K_n)^{K_p}_k.$$

The previous statement is shortly denoted by $N \to (n)^p_k$, which is the original Erdős-Rado partition arrow.

In Ramsey theory one tries to prove the validity of statement $C \to (B)^A_k$ for various combinatorial, number theoretical and geometrical structures. For a good survey on this topic, see, *e.g.*, [22] or [33].

Another question, which is intensively studied, is motivated by efforts to find, for a given A, B and k, the *minimal size* of the structure C satisfying $C \to (B)^A_k$. Denote by *$\mathcal{C}$-Ramsey number* $r_{\mathcal{C}}(A, B, k)$ the minimal size of $C \in \mathcal{C}$ which satisfies $C \to (B)^A_k$ in a fixed class $\mathcal{C}$ of structures where all the objects A, B, C are considered (we tacitly assume $C \in \mathcal{C}$ exists; otherwise we leave $r_{\mathcal{C}}(A, B, k)$ undefined).

These questions seem to be very difficult even in the simplest instances, such as the Ramsey theorem. In this case

$$2^{n/2} \leq r_{\mathcal{K}}(K_2, K_n, 2) \leq 2^{2n},$$

where $\mathcal{K}$ is the class of all complete graphs. This leads to tower function growth for numbers $r_{\mathcal{K}}(K_p, K_n, k)$.

For other structures, such as other classes of graphs, hypergraphs (with induced subgraphs and subhypergraphs), arithmetic progressions (the van der Waerden theorem), combinatorial cubes (the Hales-Jewett theorem), the situation is much less satisfactory and in most instances one is satisfied with the existence of an object C in the previously described setup, without trying to optimise its size, which seems to be extremely large for such structures.

Let $\Delta = (\delta_i;\ i \in I)$ be an integer sequence called a *type.* An *ordered relational structure S of type* Δ is a tuple $S = (X, (R_i;\ i \in I))$ where X is an ordered set and $R_i \subseteq X^{\delta_i}$ (*i.e.*, R_i is a δ_i-ary relation); we denote $V(S) = X$. A structure $S' = (X', Y')$ is a *substructure* of structure $S = (X, Y)$ if $X' \subseteq X$, $Y' \subseteq Y$, $Y' \subseteq 2^{X'}$ and X' preserves the ordering of X. A class $\mathcal{C}$ of structures is called *Ramsey class* if for every $A, B \in \mathcal{C}$ and every k there exists $C \in \mathcal{C}$ such that $C \to (B)^A_k$. Let us list few examples of Ramsey classes.

For a fixed Δ, we shall consider the class $Rel(\Delta)$ of all finite ordered relational structures of type Δ. A structure $S = (X, (R_i;\ i \in I))$ of type Δ is called *irreducible,* if for every pair $x, y \in X$ there exist $i \in I$ and $R \in R_i$ such that $x, y \in R$. Let $\mathcal{F}$ be a (possibly infinite) set of structures of type Δ. Denote by $Forb_\Delta(\mathcal{F})$ the class of all ordered structures A of type Δ which do not contain any member of $\mathcal{F}$ as a substructure (not necessarily induced).

Theorem 5.1. *(Nešetřil, Rödl [36], [38]) Let Δ be a type and let $\mathcal{F}$ be a (possibly infinite) set of irreducible structures of type Δ. Then the classes $Rel(\Delta)$ and $Forb_\Delta(\mathcal{F})$ are Ramsey.*

Jószef Beck initiated a systematic study of Ramsey numbers in a setting of combinatorial games. He showed that the game versions of Ramsey number are much easier to estimate. Particularly, for the case of the Ramsey theorem and the van der Waerden theorem, he obtained asymptotically optimal results ([5], [6], see also [7]).

5.2 Positional Ramsey games

Let us now turn a general Ramsey-type theorem into a game. (This transformation is contained already in one of the earliest papers of Ramsey theory [24] by Hales and Jewett where the authors interpreted the Hales-Jewett theorem.) We consider a structure C as a board. There are two players, I and II. The players alternately pick substructures $A' \in \binom{C}{A}$. I wins if and only if he succeeds to find $B' \in \binom{C}{B}$ such that the whole set $\binom{B'}{A}$ is claimed by him. Otherwise II wins. We call this game *the Ramsey* (A, B)*-game* on C and we denote the fact that I wins by

$$C \to (B)^A.$$

It follows by a general strategy stealing argument that I wins provided $C \to (B)^A_2$. However, this is far from necessary. To clarify this, let us denote by $r^g_{\mathcal{C}}(A, B)$ the minimal size of $C \in \mathcal{C}$ measured by $|V(C)|$ satisfying that I wins Ramsey (A, B)-game on C (provided such a C exists). It appears that in most cases we can claim that $r^g_{\mathcal{C}}(A, B)$ has a moderate size. This phenomenon was already exhibited in 1981 by Beck in a landmark paper [5] and in 2002 in [6]:

Theorem 5.2. *(Beck [5]) Consider the game version of the van der Waerden theorem and let $r^g(AP(n))$ denote the minimum size N such that Player* I *wins the game of building a arithmetic progression of length n on the set $\{1, \dots, N\}$. Then*

$$\lim_{n\to\infty} \sqrt[n]{r^g(AP(n))} = 2.$$

Theorem 5.3. *(Beck [6]) Let us consider the Ramsey (E_p, K_n)-game in the class $\mathcal{K}^p$ of all p-uniform complete hypergraphs, where E_p is a hypergraph edge, and let the board be the hypergraph $K_N \in \mathcal{K}^p$. In case $p = 2$ (graphs), if*

$$n \geq 2\log_2 N - 2\log_2 \log_2 N + 2\log_2 e - 1 + \wr(1),$$

then Breaker has an explicit winning strategy. On the other hand, if

$$n \leq 2\log_2 N - 2\log_2 \log_2 N + 2\log_2 e - \frac{10}{3} + \wr(1),$$

then Maker has an explicit winning strategy. In case $p \geq 3$, Breaker wins if

$$n \geq (p! \log_2 N)^{\frac{1}{p-1}} + \wr(1),$$

and Maker wins if

$$n \leq (p! \log_2 N)^{\frac{1}{p-1}} - \mathcal{O}(1).$$

This should be compared with the bound for Ramsey function mentioned earlier. For Theorem 5.2, let us just recall that the van der Waerden function is known to be primitive recursive (as shown first by Shelah [45]) and the bound of order $\mathcal{O}(2^{2^{2^{2^{2^{n}}}}})$ was obtained more recently by Gowers [21]. Nevertheless, the lower bound is exponential only. Thus the game Ramsey function may be drastically smaller than the Ramsey function.

5.3 Ramsey games on relational structures

We generalise the results of Beck to $\mathcal{C}$-Ramsey numbers for relational structures. The main message of these results is that the game version of $\mathcal{C}$-Ramsey numbers may be essentially smaller than the extremely large $\mathcal{C}$-Ramsey numbers. And sometimes game Ramsey numbers exist even in the situation where Ramsey-type results are not true.

Let $S = (X, (R_i;\ i \in I))$ be a structure of type Δ. The *inflation of* $x \in X$ *by a factor* k is the structure $S' = (X', (R'_i;\ i \in I))$ of type Δ defined as follows:

$$X' = \{y \in X;\ y \neq x\} \cup V_x, \quad V_x = \{y_1^x, y_2^x, \ldots, y_k^x\}, \tag{9}$$

$$R'_i = \{\{y^x, y_1, \ldots, y_{\delta_i - 1}\};\ y^x \in V_x,\ \{y, y_1, \ldots, y_{\delta_i-1}\} \in R_i\}, \quad i \in I. \tag{10}$$

The set V_x is called *multivertex*. The *inflation of* S *by a factor* k is the structure S^k such that every $x \in X$ was inflated by k.

Constructions similar to inflation are often used in combinatorics. Depending on the context they are called, for example, blowing-up, multiplication of points, or homomorphism preimages.

We show the following:

Theorem 5.4. *Let $\mathcal{C}$ be a class of structures which is closed under vertex inflation. Then for every $B \in \mathcal{C}$ and every $A \subset B$, $|V(A)| < |V(B)|$, there exists $C \in \mathcal{C}$ such that $C \to (B)^A$. Moreover, $|V(C)| \leq 2^u \cdot u \cdot |V(B)|$ where $u = \left|\binom{B}{A}\right|$. Particularly, $r^g_{\mathcal{C}}(A, B) \leq 2^u \cdot u \cdot |V(B)|$.*

The condition of inflation holds, for example, for any class $\mathcal{C}$ of structures which is determined by forbidden homomorphisms from a finite set of structures $F_1, \ldots, F_t$:

$$\mathcal{C} = Forb(F_1, \ldots, F_t) = \{G;\ F_i \not\to G,\ i = 1, \ldots, t\},$$

where by $F_i \not\to G$ we denote there does not exist a homomorphism from F_i to G. This covers, for example, class of K_k-free graphs, which is known to be a Ramsey class (and thus C satisfying $C \to (B)^A_k$ can be applied in Theorem 5.4). However, the class $Forb(F_1, \ldots, F_t)$ is Ramsey if and only if F_i are complete graphs. Thus Theorem 5.4 covers many examples when $\mathcal{C}$ fails to be a Ramsey class.

Denote by $\mathcal{G}$ the class of all undirected graphs. It is known that there is no (unordered) graph G satisfying

$$G \to (C_5)_2^{K_{1,2}},$$

and thus $r_{\mathcal{G}}(K_{1,2}, C_5, 2)$ is undefined. On the other hand, $r^g_{\mathcal{G}}(K_{1,2}, C_5)$ exists and it is $r^g_{\mathcal{G}}(K_{1,2}, C_5) \leq 800$. In fact, $r^g_{\mathcal{G}}(G, H)$ exists for any graphs G, H.

Analogously with the definition of Ramsey classes, we can thus define game Ramsey classes. A class $\mathcal{C}$ of structures is called *game Ramsey class* if for every $A, B \in \mathcal{C}$ there exists $C \in \mathcal{C}$ such that $C \to (B)^A$.

Corollary 5.5. *Let $\mathcal{C}$ be a class of structures which is closed under vertex inflation. Then $\mathcal{C}$ is a game Ramsey class.*

This should serve as a warm up to Theorem 5.4 and further examples of the structural Ramsey theorem whose game versions we shall consider. Our examples include restricted Ramsey theorems for set systems and extended versions of van der Waerden theorem (Brauer theorem).

We believe that the fact that so broad classes of structures can be proved to be game Ramsey classes is a quite surprising fact which may lead to some consequences in the model theory (similarly as Ramsey classes did, see Nešetřil [34] and Kechris, Pestov, Todorcevic [27]).

Theorem 5.4 also leads to challenging problems. Perhaps the most interesting is the question whether these results can be modified to obtain results for strong Ramsey theory games. A strong game is defined by a change of the winning criterion: the first player who achieves a monochromatic copy of structure B wins. This game may result in a draw.

The strong game is much harder to analyse and presently there is no analogy of Theorem 5.4 for strong games. Nevertheless, we show a peculiar result:

Theorem 5.6. *There exists a graph G with the following properties:*

1. *G does not contain K_4,*

2. *the first player wins strong (K_2, K_3)-game on G,*

3. *G has 9 vertices.*

This result is more interesting in its context than its proof (a case analysis, which seems to be typical for the analysis of strong games). The question of the existence of K_4-free graph G satisfying $G \to (K_3)_2^{K_2}$ was first answered by Folkman [14] and later fully solved by Nešetřil and Rödl [35]. Erdős asked whether there is such a G of size less than 10^{10}. Recently, Dudek and Rödl [12] showed that there is such a G of order 941.

5.4 Colouring Vertices

We can easily adapt Theorem 5.4 for the vertex colouring, *i.e.*, the (K_1, B)-game. However, in this case we can easily analyse even the strong vertex game:

Theorem 5.7. *Let $\mathcal{C}$ be a class of structures which is closed under inflation. Let $B \in \mathcal{C}$ and $p = |V(B)|$. Then there exists $C \in \mathcal{C}$ on $2p-1$ vertices such that Player* I *wins the strong Ramsey (K_1, B)-game on C. Moreover, the size $2p-1$ cannot be improved.*

By a *cycle* C_s we mean every hypergraph satisfying the following condition: there exists a sequence $(v_1, E_1, v_2, E_2, \ldots, v_s, E_s)$ such that all v_i and all E_i are distinct and $v_i \in V(C_s)$, $E_i \in E(C_s)$, and $v_i, v_{i+1} \in E_i$ for $i = 1, \ldots, s-1$ and $v_s, v_1 \in E_s$. For a hypergraph G, by girth(G) we mean the minimum s such that G contains a cycle C_s.

In the case when the class $\mathcal{C}$ contains only cycle-free structures, we have to use different technique. Our basic tool is the following lemma, which shows that there exist "dense" hypergraphs without short cycles. Its proof is an application of probabilistic method and it follows Erdős and Spencer [13] where the original proof can be found. We use the approach presented by Nešetřil and Rödl [37].

Lemma 5.8. *(Erdős, Spencer) For all positive integers k and s there exists a k-uniform hypergraph $G = (V, E)$, $|V| = n$, without cycles of length less than s and with $|E| > n^{1+1/s}$ edges for all n sufficiently large.*

We mention that the proof of Lemma 5.8 is not constructive, *i.e.*, it gives the desired hypergraph G by purely existential argument. We prove the following.

Theorem 5.9. *Let $\mathcal{F}$ be a set of 2-connected hypergraphs and let $\mathcal{C} = Forb(\mathcal{F})$. Let $B \in \mathcal{C}$, $p = |V(B)|$ and*

$$\ell = \max_{i=1,\ldots,t} \text{girth}(F_i).$$

Then there exists $C \in \mathcal{C}$ on $\mathcal{O}(2^{p\ell})$ vertices such that $C \to (B)^{K_1}$.

5.5 Arithmetic Progression Games

The following theorem, conjectured by Schur and proved in 1928 by Brauer [10], is an extension of the van der Waerden theorem.

Theorem 5.10. *(Brauer [10]) For a positive integer n, there exists a positive integer N such that in an arbitrary colouring of the set $[N]$ by r colours, we can find in one of the colour classes the arithmetic progression $a_0, a_0 + d, \ldots, a_0 + nd$ together with the difference d.*

For two integers $k \geq 3$ and n, we define the *arithmetic progression game with difference* on the set $S = \{1, 2, \ldots, n\}$ as follows. Maker and Breaker alternately pick elements of S. Maker wins if he picks some $(k+1)$ elements of S where the first k elements form an arithmetic progression P, and the remaining element d is the difference of P. If he is unable to pick such set, Breaker wins. By $r^g(APd(k))$ we mean the smallest n such that Maker has a winning strategy.

The arithmetic progression game (*i.e.*, based on the original van der Waerden theorem) was investigated by Beck [5] (see Theorem 5.2). In the following theorem, we generalise the proof ideas of Beck to work on arithmetic progression games with difference, and we also give a lower bound on the board size.

Theorem 5.11. *Let $k \geq 2$ be an integer. Assume Maker and Breaker play the k-term arithmetic progression game with difference. Then Maker has a winning strategy on board of size $\mathcal{O}(2^k k^3)$ and Breaker has a winning strategy on board of size $\Omega(2^{k/2}\sqrt{k})$, i.e.,*

$$\Omega(2^{k/2}\sqrt{k}) \leq r^g(APd(k)) \leq \mathcal{O}(2^k k^3).$$

References

[1] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Applied Mathematics*, 8(1):1–11, 1984.

[2] B. Alspach. Searching and sweeping graphs: a brief survey. *Matematiche (Catania)*, 59(1–2):5–37, 2006.

[3] M. F. Balcan, S. Krehbiel, G. Piliouras, and J. Shin. Near optimality in covering and packing games by exposing global information. *Arxiv preprint arXiv:1109.3606*, 2011.

[4] R. Bar-Yehuda and S. Even. A linear-time approximation algorithm for the weighted vertex cover problem. *Journal of Algorithms*, 2(2):198–203, 1981.

[5] J. Beck. Van der waerden and ramsey type games. *Combinatorica*, 1:103–116, 1981.

[6] J. Beck. Positional games and the second moment method. *Combinatorica*, 22(2):169–216, 2002.

[7] J. Beck. *Combinatorial Games: Tic-Tac-Toe Theory*. Cambridge University Press, 2008.

[8] K. Bhawalkar, M. Gairing, and T. Roughgarden. Weighted congestion games: Price of anarchy, universal worst-case examples, and tightness. *Algorithms–ESA*, pages 17–28, 2010.

[9] R. V. Book. Comparing complexity classes. *Journal of Computer and System Sciences*, 9(2):213–229, 1974.

[10] A. Brauer. Über sequenzen von potenzresten. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, pages 9–16, 1928.

[11] J. Cardinal and M. Hoefer. Selfish service installation in networks. *Internet and Network Economics*, pages 174–185, 2006.

[12] A. Dudek and V. Rödl. On the folkman number $f(2,3,4)$. *Experimental Mathematics*, 17(1):63–67, 2008.

[13] P. Erdős and J. Spencer. *Probabilistic methods in combinatorics.* Academic Press, New York, 1974.

[14] J. Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM Journal on Applied Mathematics*, 18:19–24, 1970.

[15] F. Fomin, P. Golovach, A. Hall, M. Mihalák, E. Vicari, and P. Widmayer. How to guard a graph? *Algorithmica*, 61(4):839–856, 2001.

[16] F. V. Fomin, P. A. Golovach, and J. Kratochvíl. On tractability cops and robbers game. *Proceedings of the 5th IFIP International Conference on Theoretical Computer Science (TCS 2008)*, 237:171–185, 2008.

[17] F. V. Fomin, P. A. Golovach, J. Kratochvíl, N. Nisse, and K. Suchan. Pursuing a fast robber on a graph. *Theoretical Computer Science*, 411:1167–1181, 2010.

[18] F. V. Fomin, P. A. Golovach, and D. Lokshtanov. Guard games on graphs: Keep the intruder out! *Theoretical Computer Science*, 412(46):6484–6497, 2011.

[19] F. V. Fomin and D. M. Thilikos. An annotated bibliography on guaranteed graph searching. *Theoretical Computer Science*, 399:236–245, 2008.

[20] A. S. Goldstein and E. M. Reingold. The complexity of pursuit on a graph. *Theoretical Computer Science*, 143:93–112, 1995.

[21] W. T. Gowers. A new proof of szemerédi's theorem. *Geometric and Functional Analysis*, 11:465–588, 2001.

[22] R. Graham, B. Rothschild, and J. Spencer. *Ramsey Theory*. 1980.

[23] F. Grandoni, J. Könemann, and A. Panconesi. Distributed weighted vertex cover via maximal matchings. *Computing and Combinatorics*, pages 839–848, 2005.

[24] A. W. Hales and R. I. Jewett. Regularity and positional games. *Transactions of the American Mathematical Society*, 106:222–229, 1963.

[25] N. G. Hall and D. S. Hochbaum. A fast approximation algorithm for the multicovering problem. *Discrete Applied Mathematics*, 15(1):35–40, 1986.

[26] S. Iwata and K. Nagano. Submodular function minimization under covering constraints. *Foundations of Computer Science (FOCS)*, pages 671–680, 2009.

[27] A. S. Kechris, V. G. Pestov, and S. Todorcevic. Fraïssé limits, ramsey theory, and topological dynamics of automorphism groups. *Geometric And Functional Analysis*, 15:106–189, 2005.

[28] S. Khot and O. Regev. Vertex cover might be hard to approximate to within 2-ε. *Computational Complexity*, pages 379–386, 2003.

[29] S. Khuller, U. Vishkin, and N. Young. A primal-dual parallel approximation technique applied to weighted set and vertex covers. *Journal of Algorithms*, 17(2):280–289, 1994.

[30] C. Koufogiannakis and N. Young. Greedy δ-approximation algorithm for covering with arbitrary constraints and submodular cost. *Automata, Languages and Programming*, pages 634–652, 2009.

[31] C. Koufogiannakis and N. E. Young. Distributed and parallel algorithms for weighted vertex cover and other covering problems. *ACM Symposium on Principles of Distributed Computing (PODC)*, pages 171–179, 2009.

[32] H. Nagamochi. Cop-robber guarding game with cycle robber-region. *Theoretical Computer Science*, 412:383–390, 2011.

[33] J. Nešetřil. Ramsey theory. *Handbook of Combinatorics*, 2:1332–1403.

[34] J. Nešetřil. Ramsey classes and homogeneous structures. *Combinatorics, Probability and Computing*, 14:171–189, 2005.

[35] J. Nešetřil and V. Rödl. The ramsey property for graphs with forbidden complete subgraphs. *Journal of Combinatorial Theory, Series B*, 20:243–249, 1976.

[36] J. Nešetřil and V. Rödl. Partitions of finite relational and set systems. *Journal of Combinatorial Theory, Series A*, 22:289–312, 1978.

[37] J. Nešetřil and V. Rödl. A probabilistic graph-theoretical method. *Proceedings of the American Mathematical Society*, 72:417–421, 1978.

[38] J. Nešetřil and V. Rödl. Ramsey class of set systems. *Journal of Combinatorial Theory, Series A*, 34:183–201, 1983.

[39] J. Nešetřil and T. Valla. On ramsey-type positional games. *Journal of Graph Theory*, 64(4):343–354, 2010.

[40] R. Nowakowski and P. Winkler. Vertex-to-vertex pursuit in a graph. *Discrete Mathematics*, 43(2-3):235–239, 1983.

[41] P. N. Panagopoulou and P. G. Spirakis. A game theoretic approach for efficient graph coloring. *ISAAC 2008*, pages 183–195.

[42] A. Quilliot. Some results about pursuit games on metric spaces obtained through graph theory techniques. *European Journal of Combinatorics*, 7(1):55–66, 1986.

[43] T. Roughgarden. Intrinsic robustness of the price of anarchy. *ACM Symposium on Theory of Computing (STOC)*, pages 513–522, 2009.

[44] T. Roughgarden and F. Schoppmann. Local smoothness and the price of anarchy in atomic splittable congestion games. *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 255–267, 2011.

[45] S. Shelah. Primitive recursive bounds for van der waerden numbers. *Journal of the American Mathematical Society*, 1:683–697, 1988.

[46] T. V. Thirumala Reddy, D. Sai Krishna, and C. Pandu Rangan. The guarding problem – complexity and approximation. *Proceedings of IWOCA 2009, Lecture Notes in Computer Science*, 5874:460–470, 2009.

List of My Research Publications

The thesis is mainly based on the following papers:

1. G. Piliouras, T. Valla, and L. Végh. LP-based Covering games with low Price of Anarchy. Submitted to SAGT 2012, Arxiv preprint arXiv:1203.0050.

2. R. Šámal and T. Valla. On the complexity of the guarding game. Submitted to Algorithmica.

3. R. Šámal, R. Stolař, and T. Valla. Complexity of the cop and robber guarding game. To appear in proceedings of IWOCA 2011 (International Workshop on Combinatorial Algorithms).

4. J. Nešetřil and T. Valla. On Ramsey-type Positional Games. *Journal of Graph Theory*, 64(4):343–354, 2010.

Further papers, not being a part of the thesis:

1. V. Jelínek, J. Kynčl, R. Stolař, and T. Valla. Monochromatic triangles in two-colored plane. *Combinatorica*, 29(6):699-718, 2009.

2. Z. Dvořák, R. Škrekovski, and T. Valla. Planar graphs of odd-girth at least 9 are homomorphic to Petersen graph. *SIAM Journal on Discrete Mathematics*, 22(2):568–591, 2008.

3. Z. Dvořák, R. Škrekovski, and T. Valla. Four gravity results. *Discrete Mathematics*, 307(2):181–190, 2007.

4. young Czech and Slovak researcher's contest SVOČ 2006: work titled *Ramsey theory and combinatorial games*, awarded by 3rd place in category Combinatorics.

5. T. Valla and J. Matoušek. Kombinatorika a grafy I. *ITI Series*, 2005.

6. D. Král', M. Mareš, and T. Valla. Recepty z programátorské kuchařky Korespondenčního semináře z programování - II. část. *Rozhledy matematicko-fyzikální*, 80(2):25–35, 2005.